
Heat Kernels and Critical Limits

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Summary. This article is an exposition of several questions linking heat kernel measures on infinite dimensional Lie groups, limits associated with critical Sobolev exponents, and Feynmann-Kac measures for sigma models. The first part of the article concerns existence and invariance issues for heat kernels. The main examples are heat kernels on groups of the form $C^0(X, \mathbf{F})$, where X is a Riemannian manifold and \mathbf{F} is a finite dimensional Lie group. These measures depend on a smoothness parameter $s > \dim(X)/2$. The second part of the article concerns the limit $s \downarrow \dim(X)/2$, especially $\dim(X) \leq 2$, and how this limit is related to issues arising in quantum field theory. In the case of $X = S^1$, we conjecture that heat kernels converge to measures which arise naturally from the Kac-Moody-Segal point of view on loop groups, as $s \downarrow 1/2$.

1 Introduction

Given a finite dimensional real Hilbert space \mathbf{f} , there is an associated Lebesgue measure $\lambda_{\mathbf{f}}$, a (positive) Laplace operator $\Delta_{\mathbf{f}}$, and a convolution semigroup of heat kernel measures

$$\nu_t^{\mathbf{f}} = e^{-\frac{t}{2}\Delta_{\mathbf{f}}} \delta_0 = (2\pi t)^{-\dim(\mathbf{f})/2} e^{-\frac{1}{2t}|x|_{\mathbf{f}}^2} d\lambda_{\mathbf{f}}(x). \quad (1)$$

The map $\mathbf{f} \rightarrow \{\nu_t^{\mathbf{f}}\}$ is functorial, in the sense that if $P : \mathbf{f}_1 \rightarrow \mathbf{f}_2$ is an orthogonal projection, then

$$P_* \nu_t^{\mathbf{f}_1} = \nu_t^{\mathbf{f}_2}. \quad (2)$$

More generally, given a finite dimensional Lie group \mathbf{F} , with a fixed inner product on its Lie algebra \mathbf{f} , there is an induced left invariant Riemannian metric, a normalized Haar measure $\lambda_{\mathbf{F}}$, a Laplace operator $\Delta_{\mathbf{F}}$, and a convolution semigroup of heat kernel measures,

$$\nu_t^{\mathbf{F}} = e^{-t\Delta_{\mathbf{F}}/2} \delta_1 = \lim_{N \rightarrow \infty} (\exp_*(\nu_{t/N}^{\mathbf{f}}))^N \quad (3)$$

(the power refers to convolution of measures on \mathbf{F}). The map $\mathbf{F} \rightarrow \{\nu_t^{\mathbf{F}}\}$ is functorial, in the sense that if $P : \mathbf{F}_1 \rightarrow \mathbf{F}_2$ and the induced map of Lie algebras $\mathbf{f}_1 \rightarrow \mathbf{f}_2$ is an orthogonal projection, then

$$P_* \nu_t^{\mathbf{F}_1} = \nu_t^{\mathbf{F}_2}. \quad (4)$$

In Section 2 we will recall how this construction can be generalized to infinite dimensions. The form of the linear generalization is well-known: given a separable real Hilbert space \mathbf{h} , there is an associated convolution semigroup of Gaussian measures $\{\nu_t^{\mathbf{h}}\}$. An essential complication arises when $\dim(\mathbf{h}) = \infty$, namely the $\nu_t^{\mathbf{h}}$ are only finitely additive, viewed as cylinder measures on \mathbf{h} (see [St] for orientation on this point). To overcome this, following Gross, we consider an abstract Wiener space $\mathbf{h} \rightarrow \mathbf{b}$; in this framework, the Gaussians are realized as countably additive measures $\nu_t^{\mathbf{h} \subset \mathbf{b}}$ on \mathbf{b} , a Banach space completion of \mathbf{h} . Geometrically speaking, one imagines that the support of the heat kernel diffuses into the enveloping space \mathbf{b} .

In the larger group category, the objects are abstract Wiener groups: such an object consists of an inclusion of a separable Hilbert Lie group into a Banach Lie group,

$$\mathbf{H} \rightarrow \mathbf{G}, \quad (5)$$

such that the induced map of Lie algebras $\mathbf{h} \rightarrow \mathbf{g}$ is an abstract Wiener space. With some possible restrictions, the corresponding heat kernels $\nu_t^{\mathbf{H} \subset \mathbf{G}}$ can be constructed as in the finite dimensional case, and they form a convolution semigroup of inversion-invariant probability measures on \mathbf{G} (this use of Ito's ideas in an infinite dimensional context apparently originated in [DS]). This construction is functorial, in the same sense as in finite dimensions.

In Section 3 we consider invariance properties of heat kernels. The Cameron-Martin-Segal theorem asserts that for $t > 0$, the measure class $[\nu_t^{\mathbf{h} \subset \mathbf{b}}]$ is invariant with respect to translation by $h \in \mathbf{b}$ if and only if $h \in \mathbf{h}$. In the larger group category, we conjecture that $[\nu_t^{\mathbf{H} \subset \mathbf{G}}]$ is invariant with respect to translation by $h \in \mathbf{G}$ if and only if $h \in \mathbf{H}_0$ and $Ad(h)^* Ad(h) - 1$ is a Hilbert-Schmidt operator on \mathbf{h} . The evidence in favor of this conjecture is based on deep results of Driver.

In Section 4, and in the remainder of the paper, we focus on groups of maps. Suppose that X is a compact manifold and W is a real Hilbert Sobolev space corresponding to degree of smoothness $s > \dim(X)/2$. In this context the Sobolev embedding

$$W \rightarrow C^0(X; \mathbb{R}) \quad (6)$$

is a canonical example of an abstract Wiener space. If \mathbf{F} is a finite dimensional Lie group with a fixed inner product on its Lie algebra \mathbf{f} , then

$$\mathbf{H} = W(X, \mathbf{F}) \rightarrow \mathbf{G} = C^0(X, \mathbf{F}) \quad (7)$$

is an abstract Wiener group. In this example the measure $\nu_t^{\mathbf{H} \subset \mathbf{G}}$ is determined by its distributions corresponding to evaluation at a finite number of points of

X ; these distributions involve a nonlocal Green's function interaction between pairs of points.

When $X = S^1$, $\mathbf{F} = K$, a simply connected compact group with $Ad(K)$ -invariant inner product, and $s = 1$, Driver and collaborators have proven that the measure class of $\nu_t^{\mathbf{H} \subset \mathbf{G}}$ equals the measure class of a Wiener measure. This latter measure has a heuristic expression in terms of a local kinetic energy functional

$$\exp\left(-\frac{1}{2t} \int_X \langle g^{-1} dg \wedge *g^{-1} dg \rangle\right) \prod_X d\lambda(g(\theta)) \quad (8)$$

(see [Dr2]). When $\dim(X) = 2$, $s = 1$ is the critical exponent, the heat kernels are defined for $s > 1$, and the heuristic expression (8) is the Feynmann-Kac measure for the quantum field theory with fields $g : X \rightarrow K$ (the sigma model with target K).

This motivates the following questions, considered in Sections 5 and 6: is there an analogue of (7) when $s = \dim(X)/2$, and is there a sense in which heat kernels have limits as $s \downarrow \dim(X)/2$? When $s = \dim(X)/2$, the Sobolev embedding fails (because of scale invariance), and fields cannot be evaluated at individual points. Moreover for fields with values in a curved space such as K , there is not an obvious notion of generalized function.

In the case of $X = S^1$, there is at least one possible resolution. In addition to accepting that the support diffuses outside of ordinary functions, one also has to accept that it diffuses in noncompact directions in G , the complexification of K . For various reasons (e.g. the Kac-Moody-Segal point of view on loop groups) one is led to consider, in place of (7), the hyperfunction completion

$$W^{1/2}(S^1, K) \rightarrow Hyp(S^1, G). \quad (9)$$

A generic point in the hyperfunction completion is represented by a formal Riemann-Hilbert factorization

$$g = g_- \cdot g_0 \cdot g_+, \quad (10)$$

where $g_- \in H^0(\Delta^*, 0; G, 1)$, $g_0 \in G$, $g_+ \in H^0(\Delta, 0; G, 1)$, and Δ (Δ^*) is the open unit disk centered at 0 (∞). There is a family of measures on $Hyp(S^1, G)$, indexed by a level l , having heuristic expressions for their densities involving Toeplitz determinants. Whereas heat kernels (and Wiener measures) are determined by distributions corresponding to points on S^1 , these latter measures are in theory determined by distributions corresponding to points off of S^1 ; in reality, an enlightening way to display these measures has not been found. Based on symmetry considerations, we conjecture that the heat kernels parameterized by s and t converge to a measure with level $l = 1/2t$ as $s \downarrow 1/2$.

The case of most interest is $X = \hat{\Sigma}$, a closed Riemannian surface, because of (8) and the central role of sigma models in Physics. This is intertwined with the one dimensional case, by the trace map $W^1(\hat{\Sigma}) \rightarrow W^{1/2}(S)$, when S

is a 1-manifold (space) embedded in $\hat{\Sigma}$ (a Euclidean space-time). In section 6 I have tried to point out a few mathematical facts about sigma models which might be relevant to understanding how to formulate a critical limit as $s \downarrow 1$.

Surveys of related topics, with more extensive bibliographies, include [Dr2] (for Wiener measure and heat kernels on loop groups), [A] (for energy representations), and [Ga] (for sigma models).

Notation 1 (1) *Throughout this paper, all spaces are assumed to be separable and real, unless explicitly noted otherwise.*

(2) *We will frequently encounter continuous maps of Hilbert spaces $\mathbf{h} \rightarrow \mathbf{f}$ such that the quotient Hilbert space equals the Hilbert space structure on the target. If \mathbf{f} is a subspace of \mathbf{h} itself, then such a map is an orthogonal projection. In this paper we will refer to such a map $\mathbf{h} \rightarrow \mathbf{f}$ as a projection.*

(3) *Suppose that \mathbf{h} is a Hilbert space. Let \mathcal{L}_2 denote the symmetrically normed ideal of Hilbert-Schmidt operators on \mathbf{h} . For an invertible operator A on \mathbf{h} , the two conditions*

$$AA^{-t} \in 1 + \mathcal{L}_2, \quad \text{and} \quad A - A^{-t} \in \mathcal{L}_2 \quad (11)$$

are equivalent. We let $GL(\mathbf{h})_{(\mathcal{L}_2)}$ denote the group of invertible operators satisfying these two conditions (in the same way, we can define $GL(\mathbf{h})_{(\mathcal{I})}$, for any symmetrically normed ideal \mathcal{I}).

(4) *Given a measure ν on a space X , and a Borel space Y , a map $f : X \rightarrow Y$ will be said to be ν -measureable if $f^{-1}(E)$ is ν -measureable for each Borel set $E \subset Y$. The measure class of ν will be denoted by $[\nu]$.*

(5) *Lebesgue measure will be denoted by $d\lambda$, and Haar measure will be denoted by $d\lambda_G$, for a group G .*

2 General Constructions

2.1 Abstract Wiener spaces and Gaussian measures

Suppose that \mathbf{h} is a Hilbert space. The corresponding convolution semi-group of Gaussian measures, $\{\nu_t^{\mathbf{h}}\}_{t>0}$, has the heuristic expression

$$d\nu_t^{\mathbf{h}} = (2\pi t)^{-\dim(\mathbf{h})/2} e^{-\frac{1}{2t}|x|_{\mathbf{h}}^2} d\lambda_{\mathbf{h}}(x), \quad (12)$$

where $\lambda_{\mathbf{h}}$ denotes the ‘Lebesgue measure for the Hilbert space \mathbf{h} ’. The measure $\lambda_{\mathbf{h}}$ and the expression (12) have literal meaning only when \mathbf{h} is finite dimensional.

To go beyond this, following Gross, suppose that $\mathbf{h} \rightarrow \mathbf{b}$ is a continuous dense inclusion of \mathbf{h} into a Banach space. We can consistently define a finitely additive probability measure $\nu_t^{\mathbf{h} \subset \mathbf{b}}$ on \mathbf{b} in the following way.

A cylinder set is a Borel subset of \mathbf{b} of the form $p^{-1}(E)$, where $p : \mathbf{b} \rightarrow \mathbf{f}$, $p|_{\mathbf{h}} : \mathbf{h} \rightarrow \mathbf{f}$ is a finite rank projection, and E is a Borel subset of \mathbf{f} . The set

of cylinder sets is an algebra; it is a σ -algebra if and only if $\mathbf{h} = \mathbf{b}$ is finite dimensional. The functorial property of finite dimensional Gaussian measures, (2), implies that there exists a well-defined finitely additive measure $\nu_t^{\mathbf{h} \subset \mathbf{b}}$ on the algebra of cylinder sets satisfying

$$p_* \nu_t^{\mathbf{h} \subset \mathbf{b}} = \nu_t^{\mathbf{f}}, \quad (13)$$

for all p as above.

Definition 2.1. (a) The inclusion $\mathbf{h} \rightarrow \mathbf{b}$ is called an *abstract Wiener space* if the finitely additive measure $\nu_t^{\mathbf{h} \subset \mathbf{b}}$ has a (necessarily unique) countably additive extension to a Borel probability measure on \mathbf{b} , for some $t > 0$ (and hence all $t > 0$, because $\nu_t^{\mathbf{h} \subset \mathbf{b}}(E) = \nu_1^{\mathbf{h} \subset \mathbf{b}}(E/\sqrt{t})$).

(b) A map of abstract Wiener spaces,

$$(\mathbf{h}_1 \subset \mathbf{b}_1) \rightarrow (\mathbf{h}_2 \subset \mathbf{b}_2) \quad (14)$$

is a map $\mathbf{b}_1 \rightarrow \mathbf{b}_2$ such that the restriction to \mathbf{h}_1 is a projection $\mathbf{h}_1 \rightarrow \mathbf{h}_2$.

(c) The *cylindrical Schwarz space* of $\mathbf{h} \subset \mathbf{b}$ is

$$\mathcal{S}(\mathbf{h} \subset \mathbf{b}) = \lim_p p^* \mathcal{S}(\mathbf{f}), \quad (15)$$

where the limit is a directed limit over all finite rank maps of Wiener spaces $p : \mathbf{b} \rightarrow \mathbf{f}$.

(d) The Laplace operator acting on a cylindrical Schwarz function is given by

$$\Delta_{\mathbf{h} \subset \mathbf{b}}(p^* \phi) = p^*(\Delta_{\mathbf{f}} \phi) \quad (16)$$

For a map of abstract Wiener spaces, the corresponding Gaussian semi-groups push forward.

In many ways, the restriction to Banach space completions of \mathbf{h} is artificial. However the Banach framework seems elegant and natural, in part because of the following characterization and examples.

Theorem 2.2. The inclusion $\mathbf{h} \rightarrow \mathbf{b}$ is an abstract Wiener space if and only if for each $\epsilon > 0$ there exists a finite dimensional subspace $\mathbf{f}_\epsilon \subset \mathbf{h}$ such that

$$\nu_t^{\mathbf{h} \subset \mathbf{b}}\{x \in \mathbf{b} : |px|_{\mathbf{f}} > \epsilon\} < \epsilon \quad (17)$$

for all $p : \mathbf{b} \rightarrow \mathbf{f}$ vanishing on \mathbf{f}_ϵ , where $p : \mathbf{h} \rightarrow \mathbf{f}$ is a finite rank projection. If \mathbf{b} is itself a Hilbert space, this is equivalent to the condition that the inclusion $\mathbf{h} \rightarrow \mathbf{b}$ is a Hilbert-Schmidt operator.

For a discussion of this theorem, see section 3.9 of [B].

Examples 1 (a) If \mathbf{f} is a finite dimensional Hilbert space, then $\mathbf{f} = \mathbf{h} = \mathbf{b}$ is an abstract Wiener space, and $\nu_t^{\mathbf{f}}$ is given literally by (12).

(b) Suppose that X is a compact manifold, possibly with boundary, and W is a Hilbert Sobolev space of degree of smoothness $s > \dim(X)/2$. Then the Sobolev embedding

$$W \rightarrow C^0(X; \mathbb{R}) \quad (18)$$

is an abstract Wiener space. This is closely related to the fact that $W \rightarrow W^0$ is a Hilbert-Schmidt operator precisely when s is above the critical exponent.

(c) For W as in (b), if $\mathbf{h} \rightarrow \mathbf{b}$ is an abstract Wiener space, then

$$W(X, \mathbf{h}) = W \otimes \mathbf{h} \rightarrow C^0(X, \mathbf{b}) \quad (19)$$

is also an abstract Wiener space (see 3.11.29 of [B]).

I do not know of a good reference for (b). However the basic ideas are in [CL]. A special case of (c) is the Wiener space analogue of the path functor,

$$W \otimes \mathbf{h} = W^1([0, T], 0; \mathbf{h}, 0) \rightarrow C^0([0, T], 0; \mathbf{b}, 0) \quad (20)$$

with inner product

$$\langle x, y \rangle_{W^1} = \langle \dot{x}, \dot{y} \rangle_{L^2} = \int_0^T \langle \dot{x}(\tau), \dot{y}(\tau) \rangle_{\mathbf{h}} d\tau. \quad (21)$$

To stay within the Banach category, we have choosen $T < \infty$. By letting $T \rightarrow \infty$, we obtain a Gaussian semigroup on the semi-infinite path space

$$C^0([0, \infty), 0; \mathbf{b}, 0). \quad (22)$$

For notational simplicity, we will write

$$\nu^{\mathbf{h} \subset \mathbf{b}} = \nu_1^{W \otimes \mathbf{h} \subset C^0([0, \infty), 0; \mathbf{b}, 0)}, \quad (23)$$

and we will refer to this as the Brownian motion associated to $\mathbf{h} \subset \mathbf{b}$. Given a partition $V : 0 < t_1 < \dots < t_n$, there is an evaluation map

$$Eval_V : C^0([0, \infty), 0; \mathbf{b}, 0) \rightarrow \prod_1^n \mathbf{b} : x \rightarrow (x_i), \quad x_i = x(t_i). \quad (24)$$

In terms of the coordinates x_i ,

$$(Eval_V)_* \nu^{\mathbf{h} \subset \mathbf{b}} = d\nu_{t_1}^{\mathbf{h} \subset \mathbf{b}}(x_1) \times d\nu_{t_1 - t_2}^{\mathbf{h} \subset \mathbf{b}}(x_2 - x_1) \times \dots \times d\nu_{t_n - t_{n-1}}^{\mathbf{h} \subset \mathbf{b}}(x_n - x_{n-1}). \quad (25)$$

In particular the distribution of $\nu^{\mathbf{h} \subset \mathbf{b}}$ at time t is $\nu_t^{\mathbf{h} \subset \mathbf{b}}$. The semigroup property of $\nu_t^{\mathbf{h} \subset \mathbf{b}}$ is equivalent to the consistency of these distributions for $\nu^{\mathbf{h} \subset \mathbf{b}}$ when the partition is refined.

The path space construction can be iterated. Thus given $\mathbf{h} \subset \mathbf{b}$, there is a Brownian motion, a Brownian sheet, etc., associated to $\mathbf{h} \subset \mathbf{b}$.

2.2 Abstract Wiener groups and heat kernels

Definition 2.3. (a) An abstract Wiener group is an inclusion of a separable Hilbert Lie group \mathbf{H} into a Banach Lie group \mathbf{G} such that the associated Lie algebra map $\mathbf{h} \rightarrow \mathbf{g}$ is an abstract Wiener space.

(b) A map of abstract Wiener groups is a map of Lie groups $\mathbf{G}_1 \rightarrow \mathbf{G}_2$ such $\mathbf{H}_1 \rightarrow \mathbf{H}_2$ and the induced Lie algebra map is a projection.

(c) An abstract Wiener group $\mathbf{H} \subset \mathbf{G}$ with the property that finite rank maps of Wiener groups $p : \mathbf{G} \rightarrow \mathbf{F}$ separate the topology of \mathbf{G} is called local.

(d) For a local abstract Wiener group, we define the cylindrical Schwarz space by

$$\mathcal{S}(\mathbf{H} \subset \mathbf{G}) = \lim_p p^* \mathcal{S}(\mathbf{F}) \quad (26)$$

and the (left-invariant) Laplacian acting on a cylindrical Schwarz function by

$$\Delta_{\mathbf{H} \subset \mathbf{G}}(p^* \phi) = p^* \Delta_{\mathbf{F}} \phi. \quad (27)$$

Given an abstract Wiener group $\mathbf{H} \subset \mathbf{G}$, there is an induced left-invariant Riemannian metric on \mathbf{H} , and a left-invariant Finsler metric on \mathbf{G} . These induce separable complete metrics compatible with the topologies of \mathbf{H} and \mathbf{G} .

Examples 2 (a) If \mathbf{F} is a finite dimensional Lie group, with an arbitrary inner product on its Lie algebra \mathbf{f} , then $\mathbf{F} = \mathbf{H} = \mathbf{G}$ is an abstract Wiener group.

(b) If W is as in (b) of Examples 1, given an abstract Wiener group $\mathbf{H} \subset \mathbf{G}$,

$$W(X, \mathbf{H}) \subset C^0(X, \mathbf{G}) \quad (28)$$

is a local abstract Wiener group.

(c) The construction in (b) can be adapted to local gauge transformations of a nontrivial principal bundle $P \rightarrow X$, provided that the structure group K is a compact Lie group with a fixed $\text{Ad}(K)$ -invariant inner product.

(d) There are also nonlocal examples involving infinite classical matrix groups; see [Go].

The following would be the ideal existence result.

Conjecture 2.4. For fixed $t \geq 0$, the sequence of probability measures

$$\exp_*(\nu_{t/N}^{\mathbf{h} \subset \mathbf{g}})^N \quad (29)$$

(the N -fold convolution on \mathbf{G}) has a weak limit with respect to $BC(\mathbf{G})$, bounded continuous functions. The limits, denoted $\nu_t^{\mathbf{H} \subset \mathbf{G}}$, form a convolution semigroup of inversion-invariant probability measures on \mathbf{G} . Given a map

$$P : (\mathbf{H}_1 \subset \mathbf{G}_1) \rightarrow (\mathbf{H}_2 \subset \mathbf{G}_2) \quad (30)$$

of abstract Wiener groups, $P_* \nu_t^{\mathbf{H}_1 \subset \mathbf{G}_1} = \nu_t^{\mathbf{H}_2 \subset \mathbf{G}_2}$, for each $t \geq 0$.

This conjecture is known to be true if one additionally assumes that the inclusion $\mathbf{h} \subset \mathbf{g}$ is 2-summable (see [BD]).

I have not stated this result in a standard form, and this obscures an important idea. To explain this idea, fix $t > 0$. Consider the map

$$C^0([0, \infty), 0; \mathbf{g}, 0) \rightarrow \mathbf{G} : x \rightarrow g_N(t) = e^{x_1} e^{x_2 - x_1} \dots e^{x_N - x_{N-1}}, \quad (31)$$

where $x_i = x(it/N)$. The $g_N(t)$ -distribution of the Brownian motion measure $\nu^{\mathbf{h} \subset \mathbf{g}}$ is the N -fold convolution (29), because the $x_i - x_{i-1}$ are independent (\mathbf{g} -valued) Gaussian variables with variance t/N . Thus the conjecture is equivalent to the assertion that $g_N(t)$ has a limit as $N \rightarrow \infty$, say $g(t)$, with $\nu^{\mathbf{h} \subset \mathbf{g}}$ -probability one. The latter statement implies that the heat kernel is an image of a Gaussian measure, an insight due to Ito (the standard approach goes a step further, and constructs $g(t)$, as a continuous function of t , by showing that it solves a stochastic differential equation).

Examples 3 (a) If \mathbf{F} is a finite dimensional real Lie group, with an arbitrary inner product on the Lie algebra \mathbf{f} , then for Δ the Laplacian for the left invariant Riemannian metric,

$$\nu_t^{\mathbf{F}} = e^{-t\Delta/2} \delta_1, \quad (32)$$

This measure is absolutely continuous with respect to Haar measure, and has a real analytic density.

(b) Consider the path construction

$$\mathbf{H} = W^1(I, 0; \mathbf{F}, 1) \rightarrow \mathbf{G} = C^0(I, 0; \mathbf{F}, 1) \quad (33)$$

where $I = [0, T]$, and \mathbf{h} has the norm (21) (with \mathbf{f} in place of \mathbf{h} in (20)). In this case $\nu_t^{\mathbf{H} \subset \mathbf{G}}$ is Brownian motion for \mathbf{F} , with variance t . The path space for \mathbf{G} is the double path space

$$C^0(I \times I, \{0\} \times I \cup I \times \{0\}; \mathbf{F}, 1), \quad (34)$$

and $\nu^{\mathbf{H} \subset \mathbf{G}}$ is the \mathbf{F} -valued Brownian sheet.

The Brownian sheet is relevant to Physics in the following way. Suppose that $\mathbf{F} = K$, a compact Lie group, and the inner product is $Ad(K)$ -invariant. Suppose that Σ is a closed surface with an area element. The Yang-Mills measure is the measure on the space of K -connections which has the heuristic expression

$$e^{-\frac{1}{2} \int \langle F_A \wedge * F_A \rangle} d\lambda(A) \quad (35)$$

for $A \in \Omega^1(\Sigma; \mathfrak{k})$, where F_A is the curvature for the connection $d + A$, and $d\lambda$ denotes a heuristic Lebesgue measure on the linear space of \mathfrak{k} -valued one forms. This measure can be (roughly) defined as a finitely additive measure in the following way. Let $p_t(g)d\lambda(g)$ denote the heat kernel for K . Given a triangulation of Σ , one considers the projection

$$\Omega^1(\Sigma, \mathfrak{k}) \rightarrow K^E : A \rightarrow (g_e) \quad (36)$$

where E denotes the set of edges for the triangulation, and $g_e \in K$ represents parallel translation for the connection $d + A$. The image of the Yang-Mills measure is

$$\prod_F p_{t\text{Area}(f)}(g_{\partial f}) \prod_E dg_e, \quad (37)$$

where $g_{\partial f}$ denotes the holonomy around the face $f \in F$. The fact that these measures are coherent with respect to refinement of the triangulation follows directly from the semigroup property of the heat kernel. When projected to gauge equivalence classes, this measure can be expressed as a finite dimensional conditioning of the Brownian sheet (see [Sen]).

3 Invariance Questions

The linear invariance properties of Gaussian measures are summarized by the following

Theorem 3.1. *Suppose that $\mathbf{h} \rightarrow \mathbf{b}$ is an abstract Wiener space. Let $\nu = \nu_t^{\mathbf{h} \subset \mathbf{b}}$ for some $t > 0$.*

- (a) *The subset of translations in \mathbf{b} which fix $[\nu]$ is \mathbf{h} .*
- (b) *The group of linear transformations which fix $[\nu]$ is $GL(\mathbf{h})_{(\mathcal{L}_2)}$, in the following sense: given a ν -measureable linear map $\hat{T} : \mathbf{b} \rightarrow \mathbf{b}$ which fixes $[\nu]$, $T = \hat{T}|_{\mathbf{h}} \in GL(\mathbf{h})_{(\mathcal{L}_2)}$; conversely, given $T \in GL(\mathbf{h})_{(\mathcal{L}_2)}$, there exists a ν -measureable linear map $\hat{T} : \mathbf{b} \rightarrow \mathbf{b}$ which fixes $[\nu]$ and satisfies $\hat{T}|_{\mathbf{h}} = T$.*

These results, and their history, are discussed in a very illuminating way in [B] (where one can also find expressions for the Radon-Nikodym derivatives).

Now suppose that $\mathbf{H} \subset \mathbf{G}$ is an abstract Wiener group, and $g \in \mathbf{G}$. When is

$$[(L_g)_* \nu_t^{\mathbf{H} \subset \mathbf{G}}] = [\nu_t^{\mathbf{H} \subset \mathbf{G}}], \quad (38)$$

where L_g is left translation? Necessarily $g \in \mathbf{G}_0$, the identity component (because $\nu_t^{\mathbf{H} \subset \mathbf{G}}$ is supported in the identity component). The abelian case suggests that $g \in \mathbf{H}$. Because $\nu_t^{\mathbf{H} \subset \mathbf{G}}$ is inversion invariant, translation invariance implies that $[\nu_t^{\mathbf{H} \subset \mathbf{G}}]$ is g -conjugation invariant. Thus if (38) holds for all t , this suggests that $[\nu_t^{\mathbf{H} \subset \mathbf{G}}]$ is $Ad(g)$ -invariant, and this implies that, as an operator on \mathbf{h} , $Ad(g) \in GL(\mathbf{h})_{(\mathcal{L}_2)}$. These considerations suggest the following

Conjecture 3.2. For $T > 0$ and $h \in \mathbf{G}$, $[\nu_T^{\mathbf{H} \subset \mathbf{G}}]$ is left and right h -invariant if and only if $h \in \mathbf{H}_0$ and $Ad(h) \in GL(\mathbf{h})_{(\mathcal{L}_2)}$.

Hilbert-Schmidt criteria are universal in unitary representation-theoretic questions of this type. In the next section we will consider evidence in favor of this conjecture.

Before leaving this abstract setting, I will mention two other related questions.

When $t \rightarrow \infty$, a Gaussian measure is asymptotically invariant in the following sense introduced by the Malliavins: given $\nu_t = \nu_t^{\mathbf{h} \subset \mathbf{b}}$ and $h \in \mathbf{h}$, for each $p < \infty$,

$$\int \left| 1 - \frac{d\nu_t(b+h)}{d\nu_t(b)} \right|^p d\nu_t(b) \rightarrow 0 \quad (39)$$

as $t \rightarrow \infty$ (in this linear context, this integral reduces to a one-dimensional integral, which is easily estimated). Now suppose that Conjecture 3.2 is true, and let $\nu_t = \nu_t^{\mathbf{H} \subset \mathbf{G}}$ and h satisfy the conditions in Conjecture 3.2. Is it true that for each $p < \infty$,

$$\int \left| 1 - \frac{d\nu_t(gh)}{d\nu_t(g)} \right|^p d\nu_t(g) \rightarrow 0 \quad (40)$$

as $t \rightarrow \infty$? The Malliavins proved that Wiener measure on $C^0(S^1, K)$ is asymptotically invariant in this sense (see chapter 4, Part III, of [Pil] for a quantitative version of this result, and an example of how this result is used to prove existence of an invariant measure for the loop group).

The second question is the following. Suppose that we change the inner product on \mathbf{h} in a way which (by (b) of Theorem 3.1) does not change the measure class of $\nu_t^{\mathbf{h} \subset \mathbf{g}}$:

$$\langle x, y \rangle_1 = \langle Ax, y \rangle, \quad x, y \in \mathbf{h} \quad (41)$$

where $A - 1 \in \mathcal{L}_2(\mathbf{h})$. Write \mathbf{h}_1 for \mathbf{h} with this inner product depending on A . Is $[\nu_t^{\mathbf{H}_1 \subset \mathbf{G}}] = [\nu_t^{\mathbf{H} \subset \mathbf{G}}]$, for each t ?

4 The Example of $Map(X, \mathbf{F})$

Let X denote a compact Riemannian manifold, and fix a Sobolev space (with a fixed inner product) of continuous real-valued functions

$$W \hookrightarrow C^0(X), \quad (42)$$

necessarily corresponding to some degree of smoothness $s > \dim(X)/2$. For the sake of clarity, we will suppose that ∂X is empty, and the inner product for W is of the form

$$\langle \phi_1, \phi_2 \rangle_W = \int_X (P^{s/2}(\phi_1))(P^{s/2}(\phi_2)) dV \quad (43)$$

where P denotes a positive elliptic psuedodifferential operator of order 2, e.g. $P = (m_0^2 + \Delta)$, where Δ is a (positive) Laplace type operator. We will write $G(x, y)$ for the Green's function of P^s ; thus

$$P_y^s(G(x, y)dV(y)) = \delta_x(y) \quad (44)$$

in the sense of distributions. Evaluation at $x \in X$ defines a continuous linear functional

$$\delta_x : W \rightarrow \mathbb{R} : \phi \rightarrow \phi(x); \quad (45)$$

this functional is represented by $G(x, \cdot) \in W$:

$$\phi(x) = \langle \phi, G(x, \cdot) \rangle_W. \quad (46)$$

Let \mathbf{F} denote a finite dimensional abstract Wiener group. Then

$$\mathbf{H} = W(X, \mathbf{F}) \rightarrow \mathbf{G} = C^0(X, \mathbf{F}) \quad (47)$$

is an abstract Wiener group, where $\mathbf{h} = W \otimes \mathbf{f}$, as a Hilbert space. It is local because we can evaluate at points of X .

Given a finite set of points $V \subset X$, let

$$Eval_V : C^0(X, \mathbf{F}) \rightarrow \mathbf{F}^V : g \rightarrow (g(v)) \quad (48)$$

denote the evaluation map. This is a map of abstract Wiener groups, where \mathbf{f}^V has the Hilbert space structure induced by the Lie algebra map

$$Eval_V : W(X, \mathbf{f}) \rightarrow \mathbf{f}^V. \quad (49)$$

It follows from (46) that this inner product on \mathbf{f}^V is given by

$$\langle (x_v), (y_w) \rangle = \sum_{v, w \in V} G^{v, w} \langle x_v, y_w \rangle_{\mathbf{f}}, \quad (50)$$

where $(G^{v, w})$ is the matrix inverse to the covariance matrix $(G(v, w))_{v, w \in V}$.

More generally, given an embedded submanifold $S \subset X$, restriction induces a map of abstract Wiener groups

$$C^0(X, \mathbf{F}) \rightarrow C^0(S, \mathbf{F}), \quad (51)$$

where the inner product on $W(S, \mathbf{f})$ corresponds to a degree of smoothness $s - \text{codim}(S)/2$. This is very reminiscent of functorial properties which are exploited at a heuristic level for Feynmann-Kac measures (see Section 6).

If f_1, \dots, f_N denotes an orthonormal basis for \mathbf{f} , and if $f_i^{(v)}$ denotes the left invariant vector field on \mathbf{F}^V corresponding to the v -coordinate, then the left invariant Laplacian on \mathbf{F}^V determined by the inner product (50) is given by

$$\Delta_V = \sum_{v, w \in V} \sum_{j=1}^N G(v, w) f_i^{(v)} f_j^{(w)}, \quad (52)$$

and

$$(Eval_V)_* \nu_t^{\mathbf{H} \subset \mathbf{G}} = e^{-t\Delta_V/2} \delta_1^{(\mathbf{F}^V)}. \quad (53)$$

These finite dimensional distributions determine the measure $\nu_t^{\mathbf{H}^\mathbb{C}\mathbf{G}}$. We will compare this with Wiener measure below, when $X = S^1$.

We now want to understand the content of Conjecture 3.2. Suppose that $g \in W(X, \mathbf{F})$. We need to compute the adjoint for $Ad(g)$, denoted $Ad(g)^{*w}$, acting on the Hilbert Lie algebra $W(X, \mathbf{f})$. Let $Ad(g)^t$ denote the adjoint for $Ad(g)$ acting on $L^2(X, dV) \otimes \mathbf{f}$. Then

$$\langle Ad(g)^{*w}(f_1), f_2 \rangle_{\mathbf{h}} = \int P^s f_1 Ad(g)(f_2) dV \quad (54)$$

$$= \int P^s (P^{-s} Ad(g)^t P^s)(f_1) f_2 dV. \quad (55)$$

Thus

$$Ad(g)^{*w} = P^{-s} \circ Ad(g)^t \circ P^s, \quad (56)$$

and

$$Ad(g) + Ad(g)^{*w} = (Ad(g)P^{-s} + P^{-s}Ad(g)^t)P^s. \quad (57)$$

In general this is simply a zeroth order operator. However suppose that $Ad(g)$ has values in $O(\mathbf{f})$, the orthogonal group. Then

$$Ad(g) + Ad(g)^{*w} = [Ad(g), P^{-s}]P^s, \quad (58)$$

and because of the commutator, the order drops to -1 (see [Fr]).

To formulate this in operator language, let \mathcal{L}_d^+ denote the ideal of compact operators T on \mathbf{h} satisfying

$$\sup_N \frac{1}{\log(N)} \sum_1^N s_n(T), \quad (59)$$

where $s_1 \geq s_2 \geq \dots$ are the eigenvalues of $|T|$. The fact that (58) has order -1 implies that it belongs to \mathcal{L}_d^+ . This implies the following statement.

Theorem 4.1. *Let $d = \dim(X)$.*

(a) *Suppose that $g \in W(X, \mathbf{f})$ and $Ad(g(x)) \in O(\mathbf{f})$, for each $x \in X$. Then*

$$Ad(g) \in GL(\mathbf{h})_{(\mathcal{L}_d^+)}. \quad (60)$$

(b) *Suppose that $\mathbf{F} = K$, or $\mathbf{F} = K^\mathbb{C}$, where K is a compact Lie group, and the inner product on \mathbf{f} is $Ad(K)$ -invariant. Assuming the truth of Conjecture 3.2, $[\nu_t^{\mathbf{H}^\mathbb{C}\mathbf{G}}]$ is $W(X, K)$ -invariant when $X = S^1$, and noninvariant for $\dim(X) > 1$.*

The important point is that $\dim(X) = 2$ is the critical case, where the criterion marginally fails.

We now want to consider evidence in support of Conjecture 3.2.

Suppose that $X = S^1$, and $P = m_0^2 - (\frac{\partial}{\partial \theta})^2$. For $s > 1/2$,

$$G(\theta_1, \theta_2) = G(\theta_1 - \theta_2) = \frac{1}{\pi} \sum_{n \geq 0} (m_0^2 + n^2)^{-s} \cos(n(\theta_1 - \theta_2)). \quad (61)$$

We will write ν_t^{s, m_0} for the corresponding heat kernels. When $s = 1$, we can compare ν_t^{s, m_0} with Wiener measure ω_t . This latter measure has distributions

$$(Eval_V)_* \omega_t = \prod_E p_t(g_v g_w^{-1}) \prod_V d\lambda_K(g_v), \quad (62)$$

where E denotes the set of edges (between vertices in V), and $\nu_t^K = p_t d\lambda_K$. The important point is that this is a nearest neighbor interaction, which is far more elementary than the interaction involving all pairs of points for heat kernels, as in (53).

The following summarizes some of the deep results of Driver and collaborators on heat kernels for loop groups.

Theorem 4.2. *Suppose that $X = S^1$ and $\mathbf{F} = K$, a simply connected compact Lie group with $Ad(K)$ -invariant inner product on \mathfrak{f} .*

- (a) *For $s = 1$, $[\nu_t^{s, m_0}] = [\omega_t]$.*
- (b) *For all $s > 1/2$, $[\nu_t^{s, m_0}]$ is $W(S^1, K)$ -biinvariant.*

A relatively short and illuminating discussion of (a) can be found in [Dr2]. Part (b) is a long story. For $s = 1$, part (b) follows from (a) and the relatively well-understood fact that $[\omega_t]$ is $W^1(S^1, K)$ -biinvariant. Driver gave a direct proof of (b), for $s = 1$ ([Dr1]), and the ideas were shown to extend to all $s > 1/2$ in [I].

In support of Conjecture 3.2, one can also directly analyze the Brownian motion construction ((b) of Example 2), using a stochastic Ito map (which essentially linearizes the problem).

When X is a closed Riemannian surface, we expect that the heat kernels for $W^{s, m_0}(X, K) \rightarrow C^0(X, K)$ to just miss being translation quasiinvariant, for all $s > 1$. It is interesting to ask whether there is another natural construction which yields a quasiinvariant measure. This should be compared with Shavgulidze's construction of quasiinvariant measures for $Diff(X)$, for X of any dimension (see page 312 of [B]).

5 The Critical Sobolev Exponent and $X = S^1$

For $s = \dim(X)/2$, the Sobolev embedding fails, and in general a $W^{\dim(X)/2}$ -function on X is not bounded (for references to the finer properties of generic functions with critical Sobolev smoothness, see [H]). This leads to a subtle situation. On the one hand, while $W^{\dim(X)/2}(X, \mathfrak{k})$ is a Hilbert space, unless \mathfrak{k} is abelian, it is not a Lie algebra. On the other hand, because K is assumed compact, a map $g \in W^{\dim(X)/2}(X, K)$ is automatically bounded. Consequently

$W^{dim(X)/2}(X, K)$, with the Sobolev topology (we do not impose uniform convergence), is a topological group, but it is not a Lie group (the study of the algebraic topology of spaces with topologies defined by critical Sobolev norms is nascent, see [Br]).

For nonabelian \mathfrak{k} , in general it is simply not clear how to form an analogue of (47), when $s = dim(X)/2$.

In the rest of this section we will focus on the case $X = S^1$. We will freely use facts about loop groups, as presented in [PS]. For simplicity of exposition, we will assume that K is simply connected, \mathfrak{k} is simple, and the inner product (on the dual) satisfies $\langle \theta, \theta \rangle = 2$, where θ is a long root (the abelian case is essentially trivial, but requires qualifications). The complexification of K will be denoted by G .

For each $s > 1/2$, there is a (Kac-Moody) universal central extension of Lie groups

$$0 \rightarrow \mathbb{T} \rightarrow \hat{W}^s(S^1, K) \rightarrow W^s(S^1, K) \rightarrow 0. \quad (63)$$

This extension exists for $s = 1/2$ as well (as an extension of a topological group), and this is the maximal domain for the extension. This extension is realized using operator methods in [PS]. If we fix a faithful representation $K \subset U(N)$, a loop $g : S^1 \rightarrow K$ can be viewed as a unitary multiplication operator M_g on $H = L^2(S^1, \mathbb{C}^N)$. Relative to the Hardy polarization $H = H_+ \oplus H_-$, where H_+ consists of functions with holomorphic extension to the disk,

$$M_g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (64)$$

where B (or C) is the Hankel operator, and A (or D) is the Toeplitz operator, associated to the loop g . The condition $g \in W^{1/2}$ is equivalent to $B \in \mathcal{L}_2$. The Toeplitz operator defines a map

$$A : W^{1/2}(S^1, K) \rightarrow Fred(H_+). \quad (65)$$

The lift of the left action of $W^{1/2}(S^1, K)$ on itself to $A^* Det$, the pullback of the determinant line bundle $Det \rightarrow Fred(H_+)$, induces (a power of) the extension (63). The upshot is that, from an analytic point of view, Kac-Moody theory is intimately related to the critical exponent and Toeplitz determinants.

The group $H^0(S^1, G)$ is a complex Lie group. An open, dense neighborhood of the identity consists of those loops which have a unique (triangular or Birkhoff or Riemann-Hilbert) factorization

$$g = g_- \cdot g_0 \cdot g_+, \quad (66)$$

where $g_- \in H^0(D^*, \infty; G, 1)$, $g_0 \in G$, $g_+ \in H^0(D, 0; G, 1)$, and D and D^* denote the closed unit disks centered at 0 and ∞ , respectively. A model for this neighborhood is

$$H^1(D^*, \mathfrak{g}) \times G \times H^1(D, \mathfrak{g}), \quad (67)$$

where the linear coordinates are determined by $\theta_+ = g_+^{-1}(\partial g_+)$, $\theta_- = (\partial g_-)g_-^{-1}$. The (left or right) translates of this neighborhood cover $H^0(S^1, G)$.

The hyperfunction completion, $\text{Hyp}(S^1, G)$, is modelled on the space

$$H^1(\Delta^*, \mathfrak{g}) \times G \times H^1(\Delta, \mathfrak{g}), \quad (68)$$

where Δ and Δ^* denote the *open* disks centered at 0 and ∞ , respectively, and the transition functions are obtained by continuously extending the transition functions for the analytic loop space. The group $H^0(S^1, G)$ acts naturally from both the left and right of $\text{Hyp}(S^1, G)$ (see ch. 2, Part III of [Pi1]).

There is a holomorphic line bundle $\mathcal{L} \rightarrow \text{Hyp}(S^1, G)$ with a map

$$\begin{array}{ccc} \hat{W}^{1/2}(S^1, K) & \rightarrow & \mathcal{L} \\ \downarrow & & \downarrow \\ W^{1/2}(S^1, K) & \rightarrow & \text{Hyp}(S^1, G) \end{array} \quad (69)$$

which is equivariant with respect to natural left and right actions by $\hat{H}^0(S^1, K)$, where $H^0(S^1, K)$ denotes real analytic maps into K . The line bundle \mathcal{L}^* has a distinguished holomorphic section σ_0 ; restricted to $W^{1/2}$ loops, $\mathcal{L}^{*m} = A^* \text{Det}$ and $\sigma_0^m = \det A$, the pullback of the canonical holomorphic section of $\text{Det} \rightarrow \text{Fred}(H_+)$, where m is the ratio of the \mathbb{C}^N -trace form and the normalized inner product on \mathfrak{k} . We will write $|\mathcal{L}|^2$ for the line bundle $\mathcal{L} \otimes \bar{\mathcal{L}}$; we can form real powers of this bundle, because the transition functions are positive.

Theorem 5.1. *For each $l \geq 0$, there exists a $H^0(S^1, K)$ -biinvariant measure $d\mu^{|\mathcal{L}|^{2l}}$ with values in the line bundle $|\mathcal{L}|^{2l} \rightarrow \text{Hyp}(S^1, G)$ such that*

$$d\mu_l = (\sigma_0 \otimes \bar{\sigma}_0)^l d\mu^{|\mathcal{L}|^{2l}} \quad (70)$$

is a probability measure. In particular there exists a $H^0(S^1, K)$ -biinvariant probability measure $d\mu = d\mu^{|\mathcal{L}|^0}$ on $\text{Hyp}(S^1, G)$.

This is a refinement of the main result in [Pi1] (see also [Pi2]). These bundle-valued measures are conjecturally unique. Uniqueness would imply invariance with respect to the natural action of analytic reparameterizations of S^1 . This independence of scale is the hallmark of the critical exponent.

One motivation for constructing the measure $\mu^{|\mathcal{L}|^{2l}}$ is to prove a Peter-Weyl theorem of the schematic form

$$H^0 \cap L^2(\mathcal{L}^{*\otimes l}) = \sum_{\text{level}(\Lambda)=l} H(\Lambda) \otimes H(\Lambda)^* \quad (71)$$

where the $H(\Lambda)$ are the positive energy representations of level $l \in \mathbb{Z}^+$ (This statement requires more explanation. Here we will simply say, Kac and Peterson proved an algebraic generalization of the Peter-Weyl theorem (see §1.7, Part I of [Pi1]), and we are seeking an analytic version, suitable for application to sewing rules discussed below). This remains incomplete.

Because $|\sigma_0|^2 = \det|A|^{2/m}$, restricted to $W^{1/2}(S^1, K)$, μ_l has a heuristic expression

$$d\mu_l = \frac{1}{Z} \det|A|^{2(l/m)} d\mu. \quad (72)$$

This begs two questions: (1) why do Toeplitz determinants have anything to do with (limits of) heat kernels and (2) how do we write the background μ in a way which suggests how to think about it analytically?

It is instructive to first consider the abelian case. When $K = \mathbb{T}$, the theorem is valid provided $l > 0$ (the background μ does not exist, reflecting a lack of ‘compactness’ in this flat case), and we consider identity components. An element of $Hyp(S^1, \mathbb{C}^*)_0$ can be written uniquely as

$$\exp\left(\sum_{n<0} x_n z^n\right) \cdot \exp(x_0) \cdot \exp\left(\sum_{n>0} x_n z^n\right) \quad (73)$$

where the sums represent holomorphic functions in the open disks. If (73) represents a loop $g \in W^{1/2}(S^1, \mathbb{T})_0$, then

$$\det|A(g)|^{2l} = \exp\left(-l \sum_{n>0} n |x_n|^2\right) \quad (74)$$

(the Helton-Howe formula). Consequently

$$d\mu_l = d\lambda_{\mathbb{T}}(\exp(x_0)) \prod_{k>0} \frac{1}{Z_k} \exp(-lk|x_k|^2) d\lambda(x_k), \quad (75)$$

and there is no diffusion in noncompact directions: $x_{-k} = -x_k^*$ a.e.. Since

$$d\nu_t^{s,m_0} = d\nu_{m_0^{-2s}t}^{\mathbb{T}}(\exp(x_0)) \prod_{k>0} \frac{1}{Z_k} \exp\left(-\frac{1}{2t}(k^2 + m_0^2)^s |x_k|^2\right) d\lambda(x_k), \quad (76)$$

(75) is the limit of heat kernels (with $l = 1/2t$) as $s \downarrow 1/2$, provided that we also send the mass m_0 to zero.

Now suppose that K is simply connected. Since K and its loop group fit into the Kac-Moody framework, it is natural to compare μ to Haar measure $d\lambda_K$ for K .

Fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+. \quad (77)$$

A generic $g \in K$ (or G), can be written uniquely as

$$g = lmau, \quad (78)$$

where $l \in N^- = \exp(\mathfrak{n}^-)$, $m \in T = \exp(\mathfrak{h} \cap \mathfrak{k})$, $a \in A = \exp(\mathfrak{h}_{\mathbb{R}})$, and $u \in N^+ = \exp(\mathfrak{n}^+)$. This implies that there is a K -equivariant isomorphism of homogeneous spaces, $K/T \rightarrow G/B^+$ and $l \in N^-$ is a coordinate for the

top stratum. Harish-Chandra discovered that in terms of this coordinate, the unique K -invariant probability measure on K/T can be written as

$$a^{4\delta} d\lambda(l) = \prod |\sigma_i(g)|^2 d\lambda(l) = \frac{1}{|l \cdot v_\delta|^4} d\lambda(l), \quad (79)$$

where (??) implicitly determines $a = a(gT)$ as a function of l , the σ_i are the fundamental matrix coefficients, $d\lambda(l)$ denotes a properly normalized Haar measure for N^- , 2δ denotes the sum of the positive complex roots for the triangular factorization (??), and v_δ is a highest weight vector in the highest weight representation corresponding to the dominant integral functional δ . The point of the third expression is that it shows the denominator for the density is a polynomial in l , hence the integrability of (79) is quite sensitive. It also follows from work of Harish-Chandra that for $\lambda \in \mathfrak{h}_\mathbb{R}^*$

$$\int a^{-i\lambda} d\lambda_K g = \prod_{\alpha > 0} \frac{\langle 2\delta, \alpha \rangle}{\langle 2\delta - i\lambda, \alpha \rangle} \quad (80)$$

(the right hand side is Harish-Chandra's \mathbf{c} -function for G/K). In particular this determines the integrability of powers of a : for $\epsilon > 0$,

$$\int a^{(1+\epsilon)2\delta} d\lambda(l) = \epsilon^{-(d-r)/2} < \infty, \quad (81)$$

d and r denote the dimension and rank of \mathfrak{g} , respectively. The formula (79) can be derived by calculating a Jacobian in a straightforward way. The formula (80) can be deduced from the Duistermaat-Heckman localization principle (in particular $\log(a)$ is a momentum map), using a (Drinfeld-Evens-Lu) Poisson structure which generalizes to the Kac-Moody framework (see [Pi3]).

Now consider the loop situation. Recall that θ_- is a coordinate for $H^0(\Delta^*, 0; G, 1)$ (which is similar to N^-). The loop analogue of the formula (79) leads to the following heuristic expression for the θ_- distribution of μ_l , where initially we think of θ_- as corresponding to a unitary loop $g \in W^{1/2}(S^1, K)$:

$$(\theta_-)_* \mu_l = \frac{1}{\mathcal{Z}} |\sigma_0(g)|^{2(2\dot{g}+l)} d\lambda(\theta_-) \quad (82)$$

$$= \frac{1}{\mathcal{Z}} \det|A(g)|^{2(2\dot{g}+l)/m} d\lambda(\theta_-) \quad (83)$$

where \dot{g} is the dual Coxeter number, and $d\lambda(\theta_-)$ is a heuristic Lebesgue background. The important point is the shift by the dual Coxeter number, which reflects a regularization of the ‘sum over all positive roots’. In a similar manner the Duistermaat-Heckman approach to (80) applies in a heuristic way to compute the g_0 distribution of μ_l :

$$\int a(g_0)^{-i\lambda} d\mu_l = \prod_{\alpha > 0} \frac{\sin(\frac{1}{2(\dot{g}+l)} \langle 2\delta, \alpha \rangle)}{\sin(\frac{1}{2(\dot{g}+l)} \langle 2\delta - i\lambda, \alpha \rangle)}. \quad (84)$$

The function on the right hand side is an affine analogue of Harish-Chandra's \mathbf{c} -function.

The heuristic expression (83) explains why μ_l is expected to be invariant with respect to $PSU(1, 1)$ (the Toeplitz determinant and the background Lebesgue measure are conformally invariant). It also points to a way of constructing the θ_- (or g_-) distribution of μ_l , by imposing a cutoff and taking a limit:

$$d((g_-)_*\mu_l)(\theta_-) = \lim_{N \uparrow \infty} \frac{1}{Z_{P_N}} \det|A(g(P_N\theta_-))|^{2(2g+l)/m} d\lambda(P_N\theta_-), \quad (85)$$

where $P_N\theta_-$ is the projection onto the first N coefficients, g_- is related to $P\theta_-$ by $P\theta_- = (\partial g_-)g_-^{-1}$, and g is a unitary loop having Riemann-Hilbert factorization (66). Many, but not all, of the details of this construction have been worked out. The main idea is that $\log(\det|A(g)|^2)$ is part of a momentum map. Consequently localization can be used to compute integrals involving the determinant against a symplectic volume element, on finite dimensional approximations.

The formula (84) for the g_0 distribution remains a conjecture, but it seems to point in a fruitful direction. The zero-mode g_0 roughly arises from the projection

$$H^0(\hat{\mathbb{C}}, 0, \infty; G, 1) \rightarrow Hyp(S^1, G)_{generic} \rightarrow G \quad (86)$$

More generally, given a closed Riemann surface $\hat{\Sigma}$ and a real analytic embedding $S^1 \rightarrow \hat{\Sigma}$, a G -hyperfunction induces a holomorphic G -bundle on $\hat{\Sigma}$, and consequently there is a bundle

$$H^0(\hat{\Sigma} \setminus S^1, G) \rightarrow Hyp(S^1, G) \rightarrow H^1(\hat{\Sigma}, \mathcal{O}_G), \quad (87)$$

where $H^1(\hat{\Sigma}, \mathcal{O}_G)$ denotes the set of isomorphism classes of holomorphic G -bundles. The stable points of $H^1(\hat{\Sigma}, \mathcal{O}_G)$ identify with the irreducible points of $H^1(\hat{\Sigma}, K)$, which has a canonical symplectic structure (see [AB]). This suggests the following

Question 5.2. Does $d\mu$ project to the (normalized) symplectic volume on the stable points of $H^1(\hat{\Sigma}, \mathcal{O}_G)$?

There is a natural generalization of this to include $d\mu^{|\mathcal{L}|^{2l}}$. But it is not clear how to incorporate reduction at points, as in (87).

Continuous loops have Riemann-Hilbert factorizations, and as a consequence there is a diagram

$$\begin{array}{ccc} W^s(S^1, K) & \rightarrow & C^0(S^1, K) \\ \downarrow & & \downarrow \\ W^{1/2}(S^1, K) & \rightarrow & Hyp(S^1, G) \end{array} \quad (88)$$

for each $s > 1/2$. We consider the W^s norm

$$\sum_n (m_0^2 + n^2)^{s/2} |x_n|^2, \quad (89)$$

and ν_t^{s,m_0} , the corresponding heat kernels, which we view as measures on $\text{Hyp}(S^1, G)$.

Conjecture 5.3. The measures ν_t^{s,m_0} have a limit as $s \downarrow 1/2$, and the measure class of this limit equals the measure class of $d\mu_l$, where $l = 1/2t$. When $m_0 \downarrow 0$, this limit converges to μ_l .

The intuition is simply that for $m_0 = 0$, the heat kernel should relax to a configuration which is conformally invariant, as $s \downarrow 1/2$. The existing hard evidence is slight. It is true in the abelian case. The heat kernel measure class $[\nu_t^{s,m_0}]$ is biinvariant with respect to $W^s(S^1, K)$, $s > 1/2$. It is expected that $[\mu_l]$ will be biinvariant with respect to $W^{1/2}(S^1, K)$, in the sense that the natural induced unitary representation

$$H^0(S^1, K \times K) \rightarrow U(L^2(d\mu_l)) \quad (90)$$

will extend continuously to $W^{1/2}(S^1, K)$ (this is true for the discrete part of the spectrum in (71)).

6 2D Quantum Field Theory

In two dimensions the question of how to formulate a notion of critical limit for heat kernels is possibly related to the question of how to mathematically formulate sigma models. To give the discussion some structure, I will focus on one aspect of quantum field theory (qft), following Segal.

Fix a dimension d , thought of as the dimension of (Euclidean) space-time. As in section 4 of [Se], let $\mathcal{C}_{\text{metric}}$ denote the category for which the objects are oriented closed Riemannian $(d-1)$ -manifolds, and the morphisms are oriented compact Riemannian d -manifolds with totally geodesic boundaries.

Definition 6.1. *A primitive d -dimensional unitary qft is a representation of $\mathcal{C}_{\text{metric}}$ by separable Hilbert spaces and Hilbert-Schmidt operators such that disjoint union corresponds to tensor product, orientation reversal corresponds to adjoint, and $\mathcal{C}_{\text{metric}}$ -isomorphisms correspond to natural Hilbert space isomorphisms.*

It is an interesting question to what extent this definition captures the meaning of locality in qft. Segal has recently advocated additional axioms, and a priori there could be primitive theories which are not truly local.

To my knowledge there does not exist an example of a nonfree qft which has been shown to satisfy Segal's primitive axioms in dimension $d > 2$. However, in two dimensions, the space of all theories is definitely large. From one point

of view, this space is the configuration space of string theory, and it is expected to have a remarkable geometric structure; see [Ma].

Before discussing sigma models, we will recall, in outline, how a ‘generic’ theory is constructed, using constructive qft techniques (see [Pi4]).

Fix (a bare mass) $m_0 > 0$, and a polynomial $P : \mathbb{R} \rightarrow \mathbb{R}$ which is bounded below. Given a closed Riemannian surface $\hat{\Sigma}$, the $P(\phi)_2$ -action is the local functional

$$\mathcal{A} : \mathcal{F}(\hat{\Sigma}) \rightarrow \mathbb{R} : \phi \rightarrow \int_{\hat{\Sigma}} \left(\frac{1}{2}(|d\phi|^2 + m_0^2 \phi^2) + P(\phi) \right) dA, \quad (91)$$

where $\mathcal{F}(\hat{\Sigma})$ is the appropriate domain of \mathbb{R} -valued fields on $\hat{\Sigma}$ for \mathcal{A} . A heuristic expression for the $P(\phi)_2$ -Feynmann-Kac measure is

$$\exp(-\mathcal{A}(\phi)) \prod_{x \in \hat{\Sigma}} d\lambda(\phi(x)). \quad (92)$$

In this two dimensional setting, there is a rigorous interpretation of (92) as a finite measure on generalized functions,

$$\frac{1}{\det_{\zeta}(m_0^2 + \Delta)^{1/2}} e^{-\int_{\hat{\Sigma}} P(\phi) :_{C_0} d\phi_C}, \quad (93)$$

where $C = (m_0^2 + \Delta)^{-1}$, $d\phi_C$ is the Gaussian probability measure with covariance C (corresponding to the critical Sobolev space $W^1(\hat{\Sigma})$), $\int : P(\phi) :_{C_0}$ denotes a regularization of the nonlinear interaction, and \det_{ζ} denotes the zeta function determinant.

Suppose that S is a closed Riemannian 1-manifold. Given an inner product for $W^{1/2}(S)$, there is an associated Gaussian measure on generalized functions. We only need the measure class, and hence we only require that the principal symbol of the operator defining the inner product is compatible with length on S . Associated to this measure class, say $\mathcal{C}(S)$, there is a Hilbert space $\mathcal{H}(S)$, the space of half-densities associated to $\mathcal{C}(S)$ (see the appendix to [Pi4]).

Suppose that Σ is a Riemannian surface with totally geodesic boundary S . We consider the closed Riemannian surface $\hat{\Sigma} = \Sigma^* \circ \Sigma$ obtained by gluing Σ to its mirror image Σ^* along S . Because $d\phi_C$ corresponds to the critical exponent $s = 1$, typical configurations for the Feynmann measure (93) are not ordinary functions. However typical configurations are sufficiently regular so that it makes sense to restrict them to S . Consequently the Feynmann measure can be pushed forward to a finite measure on generalized functions along S . Essentially because the underlying map of Hilbert spaces is the trace map $W^1(\hat{\Sigma}) \rightarrow W^{1/2}(S)$, this finite measure lands in the measure class $\mathcal{C}(S)$. By taking a square root (which is required because Σ is half of $\hat{\Sigma}$), we obtain a half density $\mathcal{Z}(\Sigma) \in \mathcal{H}(S)$.

Theorem 6.2. *The maps $S \rightarrow \mathcal{H}(S)$ and $\Sigma \rightarrow \mathcal{Z}(\Sigma)$ define a representation of Segal’s category.*

This construction extends to vector space-valued fields in a routine way. But severe problems arise for qfts with classical fields having values in non-linear targets.

Consider the sigma model with target K . A naive idea is to use heat kernels, in place of Gaussians, as backgrounds for an analogous construction. This introduces a new element: we must nudge $s > 1$, then take a critical limit. From this point of view, the renormalization group (see section 3 of [Ga]) should be a prescription for how to take this limit.

We will augment the sigma model action to include a topological term, the Wess-Zumino-Witten ‘B-field’. Ignoring s -regularization temporarily, the above outline reads as follows: we represent the Feynmann-Kac measure

$$\exp\left(-\frac{1}{2t} \int_{\hat{\Sigma}} \langle g^{-1} dg \wedge *g^{-1} dg \rangle + 2\pi i l WZW(g)\right) \prod_x d\lambda_K(g(x)) \quad (94)$$

as a density against a critical heat kernel background; we use the naturality of heat kernels to push the Feynmann-Kac measure forward along a map of generalized abstract Wiener spaces, induced by the trace map $W^1(\hat{\Sigma}, K) \rightarrow W^{1/2}(S, K)$,

$$(W^1(\hat{\Sigma}, K) \subset?) \rightarrow (W^{1/2}(S, K) \subset Hyp(S, G)); \quad (95)$$

and we obtain a Hilbert space of half-densities, and a vector, by taking the positive square root of the pushforward measure.

The point of introducing the WZW term is that when the level l is a positive integer, and $t = 1/2l$, this (WZW_l) model is solvable in many senses (see chapter 4 of [Ga], and [We]). It is believed that the model satisfies Segal’s axioms (see the Foreword to [Se]). There also is the belief, far more speculative, that there should be a one parameter family of theories which interpolates between WZW_l and the original sigma model. This deformation intuitively arises by letting $t \downarrow 0$ in the action, and is related to the expected emergence of a scale parameter in the process of regularizing the action (94), using for example heat kernels for $s > 1$, and then taking a limit as $s \downarrow 1$.

There are strong points of contact between our outline and what is known about the WZW_l model: The Hilbert space of the WZW_l model is given by the right hand side of (71). The vacuum of the theory (the vector corresponding to a disk) is $\det(A)^l$, which is a “holomorphic square root” of $d\mu_l$ (this is far more complicated than the positive square root for $P(\phi)_2$). The proof of sewing (in the holomorphic sectors, involving the WZW_l modular functor, see page 468 of [Se]), ultimately hinges on a generalized Peter-Weyl theorem (our conjectural analytical version (71) would fit perfectly with Segal’s global approach).

The bare sigma model is believed also to be solvable, but in a completely different sense: in terms of scattering (see [ORW]).

This raises the following questions: Is there a reasonable way to regularize (94) with respect to heat kernels (for $s > 1$)? Is there a renormalization group

prescription for understanding some aspect of the critical limit $s \downarrow 1$? There is possibly an important hint arising from work of Uhlenbeck (see [U]). The classical theory corresponding to (94) is ‘completely integrable’, for all values of t and l . This involves reformulating the classical equations in terms of the connection one-form $A = g^{-1}dg$ and a spectral parameter (which evolves into the deformation parameter at the quantum level). In considering the hyperfunction completion in one dimension, we abandoned unitarity. In two dimensions it apparently is necessary to relax unitarity and the zero-curvature condition.

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